Capital Reallocation and Private Firm Dynamics

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ABSTRACT

We develop a new theory of firm dynamics and capital reallocation in private firms and use it to study the taxation of business income, capital, and capital gains on business transfers. Central to our theory is the fact that certain intangible assets—like customer-bases and trade names—are specific to a business and thus not available in rental markets. Owners grow their businesses by investing in these assets themselves or by purchasing a group of assets that constitute an existing business. Given the nature of this exchange, capital is gradually transferred to more productive owners. Despite the gradual reallocation and the consequent dispersion in marginal product, the equilibrium we analyze is efficient. The theory is disciplined with U.S. administrative tax filings of privately-held S corporations. A unique feature of our data is that it includes the price and the counterparties involved in asset acquisitions. This data allows us to calibrate the production and investment technologies for otherwise unobservable intangible capital. Introducing business taxation, we study the implications for firm entry, investment, and capital transfers. We find taxes on capital gains to be the most distortive when compared to other business taxes, primarily because of the impact on business entry, and taxes on income to be the least distortive.

Key words: business transfers, capital allocation, firm dynamics, capital taxation

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1 Introduction

A significant portion of value in U.S. private businesses comes from self-created intangible assets, such as customer bases, trademarks, and going concern value.[1] These assets are rarely rented and are typically sold together when owners relocate, retire, or pursue new ventures. Because such transactions are infrequent and their details are not publicly observed, little is known about the investments that active private businesses make–despite their contribution to over half of all U.S. business income and their central role in discussions on productivity, wealth inequality, and tax policy. This paper introduces a new theory of firm dynamics that captures essential features of these business assets. Our theory is informed by measurements from administrative data on business tax filings from the Internal Revenue Service (IRS), with a key innovation being the inclusion of data on valuations and characteristics of counterparties involved in the transfer of business assets.[2] Using this model, we provide theory-based estimates of the dispersion in marginal products of capital, returns, and valuations for ongoing businesses and reexamine the classic question of how to tax business income and wealth.

Our environment is neoclassical in the spirit of Lucas (1978) and Hopenhayn (1992)—modified to include features that make it appropriate to study firm dynamics and capital allocation for private business. Goods and services are produced with *nontransferable* capital that cannot be bought or sold, *transferable* capital that can be bought and sold but not rented, and *external* factors that are rented on spot markets. The nontransferable capital captures business owner's productivity or ability, which can change over time but is inalienable. The transferable capital stocks are business assets mentioned above, that is, built by the owners and observed only when the business is sold. The external factors would in practice include employee time, physical capital, and materials. Firms in the model can grow in two ways: through costly internal investment and through purchases of other businesses that might take time. To capture the nature of business transfers, we incorporate an indivisibility in bilateral trades, effectively assuming that businesses are sold as a unit in pairwise meetings. Owners that sell can restart another business or work as employees.

Our trading protocol implies that business capital is gradually traded upwards with owners

[1] See Bhandari and McGrattan (2021) and IRS administrative data in Section 4.

[2] This work is part of a Joint Statistical Research Program Project of the Statistics of Income Division at the IRS.

that have low marginal product of capital selling to those with higher marginal product. Despite the indivisibilities in capital exchange, we show that the equilibrium allocation of capital in our model is efficient and the set of trades form pairwise stable matches. Per-unit prices vary across sales and depend on the quantity of capital sold. Price dispersion in this model is not indicative of misallocated resources.

Data from U.S. administrative tax filings of S corporations are used to calibrate the model. S corporations are private, pass-through entities and, unlike C corporations or partnerships, face ownership restrictions: their owners must be individuals and cannot be other businesses. By combining business and individual tax forms for each owner, we construct longitudinal panels that span both the business and owner life cycles. This data is particularly useful for calibrating traditional aspects of firm dynamics, such as the distribution and processes related to owners' productivity, shares of rentable inputs, and depreciation rates. A unique aspect of our calibration is the integration of revenue and expenditure data with information on business asset acquisitions. On Form 8594, which is filed by both buyers and sellers, taxpayers allocate the business purchase price across various asset categories, including marketable securities, fixed assets, and intangible assets.[3] This last category constitutes the largest share of the transferred value. Using Form 8594, we derive two key metrics: the relative size of buyers and sellers and the ratio of the price paid for a business to the size of the business sold. We demonstrate that these metrics are crucial for identifying model parameters governing the output elasticity of intangible capital and the cost of investment in intangibles. For the type of business capital that we are concerned with, these parameters cannot be determined through traditional methods that rely solely on direct observations of investment expenditures or capital use and retirements.

The model has implications for measures of capital misallocation and business wealth. As we noted above, there is no misallocation per se, but the model predicts significant dispersion in marginal products of capital due to the nature of business technologies and transfers. In our baseline, owners have opportunities to sell or buy once per year and, in that case, the standard deviation of the logarithm of the marginal product of capital is 57 percent.[4] If we assume monthly

[3]Included in intangible assets are Section 197 assets: customer- and information-based intangibles, non-compete covenants, licenses and permits, franchises, trademarks, trade names, workforce in place, business books and records, processes, designs, patterns, as well as goodwill and going concern value. This information is required to determine capital gains for sellers and asset bases for amortization by buyers.

[4]Broker data from Pratt's Stats (now DealStats) show close to one year spells between hiring the broker and the

access to capital markets, the standard deviation falls but remains high at 30 percent. With no trade allowed, it is 90 percent. The fact that there is significant dispersion in all cases highlights the critical role of indivisibility in trades.

The model has predictions for two familiar but different measures of business wealth. The first measure is the value of transferable wealth or an answer to: What is the value of the business if sold today? This value is often reported in surveys (for example, the Survey of Consumer Finances) and it is relevant for analyzing taxation of realized capital gains. For our baseline parametrization, the share of wealth that is transferable is large: 27 percent on average. The second measure of business wealth is the total value of the ongoing concern that generates a flow of dividends to owners over the business life. The total value includes not only the value of the intangible assets that can be transferred, but also the owners' inalienable productivity. The total value is the appropriate measure of business wealth owned by the entrepreneur, hence it is relevant in the context of a wealth tax. For our baseline parametrization, the total value is 2.21 times private sector value-added.

The primary analysis in this paper compares the impacts of taxing business net income, capital value, and capital gains following a sale of the business. We assess the effects of raising a fixed amount of revenue per period in a steady-state economy. Our main finding reveals that taxing business income generates a smaller welfare loss compared to taxing either capital value or capital gains. While all three tax policies distort entry, investment, and capital reallocation, their incidence varies significantly. Business income taxes primarily affect high-productivity owners who earn the most, but these taxes do not substantially deter entry, as high-productivity owners are likely to enter regardless. Regarding investment, taxes on business income are broader in scope than taxes on business sales, leading to comparatively less distortion for a given revenue target. Income taxes are also less disruptive to trade, resulting in lower overall distortion. In contrast, taxes on capital gains primarily affect low-productivity owners who are most likely to sell. Owners with low productivity are marginal entrants and capital gains taxes discourages their investment. In terms of capital reallocation, capital gains taxes are the most distortive, creating a "lock-in" effect, whereby capital remains with less productive owners due to discouraged trade. Annual taxes on the assessed value of capital create distortions that fall between those caused by business income taxes and capital gains taxes. Like income taxes, taxes on capital value are broad-based although

business sale.

the incidence of these taxes falls mainly on owners with medium levels of productivity and high stocks of capital. Like capital gains taxes, they discourage entry and investment more than business income taxes. By effectively redistributing resources from high-capital, low-productivity owners to low-capital, high-productivity owners, taxes on capital value reduce the dispersion of the marginal product of capital. Yet, this reduction in measured misallocation is modest and is outweighed by the distortions in entry and investment.

1.1 Related Literature

Our paper relates to the extensive body of work studying firm dynamics, productivity, and the allocation, valuation, and taxation of business capital.

In the seminal work by Hopenhayn (1992), a theory of firm dynamics is proposed based on stochastic productivity with perfectly divisible, competitively-traded factor inputs. Subsequent work has introduced financial constraints and adjustment costs in input markets. On the theoretical side, we retain much of the neoclassical spirit of Hopenhayn's framework but introduce technological and market-specific features relevant for most business capital currently used in production. In particular, we model capital assets as indivisible and non-rentable and the sale of a business as a transfer of a group of assets. In considering the indivisible nature of the trade, we are also building on Holmes and Schmitz (1990), although relative to their work, we include stochastic productivity, occupational choice, and most importantly, a trade-off between investing and trading business capital.

The model Hopenhayn (1992) developed has been used to measure sources of productivity differentials and to estimate investment technologies, although the scope of previous work has been narrow due to data limitations. Most measurement relies on census-based surveys of manufacturing plants. (See Cooper and Haltiwanger (2006) and Hsieh and Klenow (2009).) Manufacturing is a small share of overall business activity and a very small share of private business activity. Furthermore, much of the empirical evidence on firm dynamics pertaining to plant and equipment is not applicable to intangible assets in private businesses, as the latter is mostly unobservable.[5]

[5] Attempts have been made to measure adjustment costs using firm-level accounting data—for example, David and Venkateswaran (2019) with publicly-traded firms or Boar et al. (2022) with private non-manufacturing firms—but these data also miss the business capital of interest here. Furthermore, for large firms that operate across various locations and sell multiple products, only consolidated financial data are available.

To address this gap, we shed light on the life-cycle dynamics of private businesses by disciplining our theory with administrative longitudinal data from IRS tax filings on revenues, expenditures, and—essential to our analysis—valuations of business capital when transferred.

We also contribute to the related debate on the sources of capital misallocation. Much of the literature has focused on the role of regulatory, financial, or informational constraints. (See Hsieh and Klenow (2009), Asker et al. (2014), and David and Venkateswaran (2019).) One contribution of our paper is to provide a theory-driven measure of dispersion in the marginal products of capital for private businesses solely induced by the market structure for capital: all assets in a business are sold as a unit with terms of trade settled in pairwise meetings. Since private business capital is not observable, no empirical counterpart of our estimate exists. To put our results in context, we obtain a dispersion in marginal product that is about half as large as common estimates for physical capital in U.S. firms.

Estimates of business value for traded and non-traded private firms have been inputs to the growing literature on measuring private business wealth. Most studies in this area rely on structural models of entrepreneurship guided by survey data (see Cagetti and De Nardi (2006)) or non-structural estimation methods such as the capitalization of income flows (see Saez and Zucman (2016) and Smith et al. (2023)). Our approach to measuring business value differs in that it leverages theoretically grounded valuation concepts and primitives, informed by detailed data on business sales and the income statements of the buyers and sellers of businesses.

We contribute to the public finance literature that studies taxation of capital and the returns to capital. In standard settings with perfect financial markets, the classical uniform commodity taxation result of Atkinson and Stiglitz (1976) implies zero tax rates on capital values and returns to capital. In the absence of perfect financial markets, Aiyagari (1995) and Guvenen et al. (2023) argue for wealth taxation to address two ills: excessive savings in liquid financial wealth due to uninsurable idiosyncratic risk and misallocation due to borrowing constraints with heterogeneous returns to business capital.[6] Chari et al. (2003) is one of the few studies examining taxation of business transfers, which they analyze in the context of the Holmes and Schmitz (1990) model. Similar to Holmes and Schmitz (1990), our model incorporates a clear notion of a business transfer. Our model also features business capital investment and, because unobserved intangible investments are

[6] For studies on business income taxation, see Kitao (2008), Meh (2005), Boar and Midrigan (forthcoming), Bhandari and McGrattan (2021), and Bruggemann (2021).

non-deductible, a natural divergence from the classical Atkinson and Stiglitz (1976) prescription. In addition, our trading technology induces persistent differences in the marginal product of capital across owners. These features motivate our exploration of second-best approaches to raising revenue through a variety tax instruments: business income, capital, and capital gains from business transfers.

Finally, our work is connected to the literature on mergers, acquisitions, and the sale of business capital that uses models of random search with bargaining or directed search with one-sided heterogeneity to ensure tractability. (See, most notably, David (2021), Ottonello (forthcoming), Guntin and Kochen (2020), and Gaillard and Kankanamge (2020).) Unlike this literature, we model business sales as transactions in a frictionless, decentralized market. Building on tools from the matching literature (Choo and Siow (2006), Galichon et al. (2019)), we solve for the equilibrium set of prices, demonstrating that it leads to an efficient allocation. We find this efficiency property appealing as it allows us to isolate the dispersion in marginal products generated solely by the indivisibility of capital in private businesses. Furthermore, our focus is different: we use our model to generate predictions for the distribution of marginal products of capital, valuations, and returns for private businesses and derive implications for tax policy.

The rest of the paper is organized as follows. Section 2 details the environment, including timing of events, descriptions of problems solved by business owners, and a definition of a recursive equilibrium. A characterization of the equilibrium and the solution algorithm are provided in Section 3. In Section 4, we document statistical properties of U.S. firm-level data that guide the calibration described in Section 5. In Section 6, we document the model's predictions for the dispersion of marginal products of capital and for business wealth. In Section 7, we assess the impacts of business taxation. Section 8 concludes.

2 Theory

The economy is populated by a unit measure of individuals that can choose to run a business or work in paid-employment. Business owners are endowed with a technology that produces consumption goods. The production inputs differ in their divisibility and transferability. The first input is entrepreneurial productivity or skill, z, which is nontransferable but evolves stochastically. The second input is intangible business capital—or simply "capital"—which we denote by k. Capital is accumulated through costly investments, is transferable via stochastic access to a capital market, but is not divisible or rentable. The remaining inputs are external factors b and n that are perfectly divisible and rentable. Factor b are fixed assets such as equipment and office space in commercial buildings. Factor n is labor. The decision to switch occupations is made continuously. Details of these actions are provided next, followed by the owner's dynamic program, and a definition of a stationary recursive equilibrium.

2.1 Environment

Production. Let $s \in \mathcal{S}$ denote a pair (z, k) and use $z(s)$ and $k(s)$ to denote the first and the second component of s, respectively. Output is produced using the technology

$$y(s, b, n) = z(s)\, k(s)^{\alpha}\, b^{\beta} n^{\gamma}. \tag{1}$$

Investment. The investment technology for capital is modeled as a cost function $c(\theta)$, where c' and c'' are strictly positive. An owner incurs cost $c(\theta)$ to invest θ and accumulate additional business capital. Specifically, the change in capital over an interval of length dt is equal to

$$dk = \theta - \delta_k k,$$

where δ_k is the depreciation rate of capital.

There is a perfectly elastic supply of equipment and office space b at an exogenous rental rate. We make this assumption for the sake of expositional simplicity. For a specific value of the rental rate, the equilibrium allocation is the same as in a setting in which a competitive mutual fund sector endowed with a linear investment technology owns the fixed assets and rents them to business owners.

Productivity process. Productivity z follows the exogenous stochastic process

$$dz = \mu(z)dt + \sigma(z)\sqrt{dt}d\mathcal{W},$$

where $\mathcal{W}$ is a standard Wiener process with $d\mathcal{W} \sim \mathcal{N}(0,1)$. Importantly, z only changes while the individual is a business owner, and it is fixed while working.

Entry, exit, and occupational choice. Entry into and exit from the economy occurs at Poisson rate ψ_{dem}. Newborns draw a state $s \sim G(s)$ and decide whether to become workers or business owners. Workers and owners have an option to switch occupation at Poisson rate ψ.

Market structure. The rental markets for labor, n, and fixed assets, b, are perfectly competitive with unit costs w and r, respectively. The market structure for capital departs from the neoclassical framework in three dimensions: time to trade, bilateralness, and indivisibility. An owner with state s accesses the capital market at Poisson rate η. We refer to this intermittent access as *time to trade*. Once in the market, an owner faces a price-quantity menu denoted by $\{p^m(s,\tilde{s})\}_{\tilde{s}\in S}$ and $\{k^m(s,\tilde{s})\}_{\tilde{s}\in S}$. Consider an owner with state s, who is deciding on a trade with another owner that has state $\tilde{s}$. Owner s would pay $p^m(s,\tilde{s})$ to the trading partner $\tilde{s}$ and exit the trading stage with capital level $k^m(s,\tilde{s})$. The functions $k^m : \mathcal{S}^2 \to K$ and $p^m : \mathcal{S}^2 \to \mathcal{R}$ are determined as part of an equilibrium that we define later.[7] We refer to the ability to trade with only one partner at a time as *bilateralness*. For the allocation of capital within a match, we impose that an owner can either sell their entire capital stock, buy the entire capital stock of their trading partner, or trade no capital at all. This assumption amounts to the following restrictions on k^m. For all pairs $(s,\tilde{s}) \in \mathcal{S}^2$,

$$k^m(s,\tilde{s}) \in \{k(s)+k(\tilde{s}), k(s), 0\} \tag{2}$$

$$k^m(\tilde{s},s) + k^m(s,\tilde{s}) \leq k(s) + k(\tilde{s}) \tag{3}$$

We refer to restriction (2) as *indivisibility*. This restriction captures a key feature of our model, namely, that the reallocation of capital across owners in bilateral trades occurs in a "lumpy" fashion.

Preferences. Owners and workers are risk-neutral and discount the future at rate ρ. We made this assumption because it enables us to isolate the effects of capital indivisibility on capital allocation and to analyze the efficiency properties of the equilibrium.[8]

[7] We omit the explicit dependence of individual choices and values on $\{p^m, k^m\}$ when it is clear from the context.

[8] We leave the task of extending the framework to incorporate a theory of concentrated ownership and limited borrowing for future research.

2.2 Recursive Formulation

Let $V : S \to \mathcal{R}^+$ denote the value of an owner. Let $\lambda : \mathcal{S} \to \Delta(\mathcal{S})$ be a measure over the set $\mathcal{S}$ that describes the probability that type s is matched to an owner of type $\tilde{s}$. Let $V_{trade}(;\lambda) : \mathcal{S} \to \mathcal{R}^+$ be the owner's gains from trade for a given λ. Let $W : \mathcal{R}^+ \to \mathcal{R}^+$ be the value of being a worker at wage w. Given functions $\{p^m, k^m\}$, the owner value solves the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}(\rho + \psi_{\text{dem}})\, V(s) = \max_{b,n,\theta,\lambda}\; & z(s)k(s)^\alpha b^\beta n^\gamma - rb - wn + \partial_k V(s)\,(\theta - \delta_k k) - c(\theta) \\ & + \partial_z V(s)\mu(z) + \frac{1}{2}\partial_{zz}V(s)\sigma(z)^2 + \psi\,\{W(w) - V(s)\}^+ + \eta V_{trade}(s;\lambda).\end{aligned} \tag{4}$$

The term on the left-hand side is the annuitized value of being an owner of type s. The right-hand side includes flow output net of the wage bill and the rental cost of fixed assets, the gain from capital investment net of the cost of investment, the changes in value induced by the evolution of productivity z, the option value of exiting, and the expected gains from trade from accessing the market for capital, V_{trade}.

The last term of the HJB in equation (4) is absent from traditional firm dynamics models, and we discuss it next. Consider a given price quantity menu $\{p^m(s,\tilde{s}), k^m(s,\tilde{s})\}_{\tilde{s}\in S}$. Define $v(s,\tilde{s})$ as the value for firm type s after trade with $\tilde{s}$:

$$v(s,\tilde{s}) \equiv V\left(z(s), k^m(s,\tilde{s})\right) - p^m(s,\tilde{s}). \tag{5}$$

Note that by allowing λ to be a measure over $\mathcal{S}$, we allow the owner to mix over the set of trading partners. The gains from trade are then given by

$$V_{trade}(s) = \max_{\lambda(s,\cdot):\sum_{\tilde{s}}\lambda(s,\tilde{s})\le 1} V_{trade}(s;\lambda), \tag{6}$$

where

$$V_{trade}(s;\lambda) \equiv \int \{v(s,\tilde{s}) - V(s)\}\,\lambda(s,\tilde{s})\,d\tilde{s}. \tag{7}$$

The inequality in the constraint is strict whenever the owner does not trade with positive probability.

The value of being a worker, $W(w)$, is given by the present value of wages until the exogenous stochastic time of death. We assume that the productivity as owner z does not change unless actively running a business. An immediate consequence is that once a particular individual decides to be a worker, the choice is never overturned. Therefore, the value of being a worker is simply

$$W(w) = \frac{w}{\rho + \psi_{\text{dem}}}. \tag{8}$$

The entry and exit decisions into business ownership $\iota(s) \in \{0, 1\}$ are given by

$$\iota(s) = \{V(s) - W(w) > 0\}. \tag{9}$$

2.3 Equilibrium

We next describe the law of motion for the distribution of owners induced by the policy functions.

Let $\phi \in \Delta(\mathcal{S})$ be the distribution over owner types and let m be the mass of owners. Let operator $\mathcal{A}_{(\theta,\iota,\lambda)}$ be the infinitesimal generator associated with the value function and $\mathcal{A}^*_{(\theta,\iota,\lambda)}$ be the conjugate of $\mathcal{A}_{(\theta,\iota,\lambda)}$.

$$\begin{aligned}\mathcal{A}_{(\theta,\iota,\lambda)}V[s] = {} & \partial_k V(s)(\theta - \delta_k k) + \partial_z V(s)\mu(z) + \frac{1}{2}\partial_{zz}V(s)\sigma(z)^2 \\ & + \eta \int \{V(z(s), k^m(s,\tilde{s})) - p^m(s,\tilde{s})\} \lambda(s,\tilde{s})\, d\tilde{s} + \psi \{W(w) - V(s)\}^+\end{aligned}$$

The law of motion of ϕ and the mass m are described by

$$\dot{\phi} = \mathcal{A}^*_{(\theta,\iota,\lambda)}\phi + \left(\frac{\psi_{\text{dem}}}{m}\right)\iota dG \tag{10}$$

$$\dot{m} = \psi_{\text{dem}} \int \iota(s) dG(s) - m\left(\psi_{\text{dem}} + \psi \int (1 - \iota(s))\phi(s)\right). \tag{11}$$

The first term on the right-hand side of (11) is the entry flow due to the choice of becoming an owner upon entering the economy. The second term is the exit flow either from the economy or from the entrepreneurial sector—and into the labor market.

We are now ready to define an equilibrium.

Definition 1. A *stationary recursive equilibrium* is given by (i) value functions $V : \mathcal{S} \to \mathcal{R}^+$ and

$W : \mathcal{R}^+ \to \mathcal{R}^+$; (ii) policy functions $n : \mathcal{S} \to \mathcal{R}^+$, $b : \mathcal{S} \to \mathcal{R}^+$, $\theta : \mathcal{S} \to \mathcal{R}^+$, $\lambda : \mathcal{S} \to \Delta(\mathcal{S})$, $\iota : \mathcal{S} \to \{0, 1\}$; (iii) wage, w; (iv) price quantity menus for capital $p^m : \mathcal{S}^2 \to \mathcal{R}$ and $k^m : \mathcal{S}^2 \to K$; (v) a mass of firms m and a probability distribution over owner types $\phi \in \Delta(\mathcal{S})$ and such that:

- Given $\{k^m(\cdot), p^m(\cdot)\}$, the value function for owners and workers $\{V(\cdot), W(\cdot)\}$ solve Bellman equations (4) and (8).
- The policy function for investment, trade, and rentable inputs solve the maximization problem in equation (4). The decision to become an owner satisfies (9).
- The labor market clears

$$m + m \int n(s)\, dG(s) = 1. \tag{12}$$

- The trading arrangements are feasible, that is, for all pairs $(s, \tilde{s}) \in \mathcal{S}^2$ with each having a positive density under ϕ the function k^m satisfies (2)-(3). The trading policies are mutually consistent, that is,

$$\lambda(s, \tilde{s})\phi(s) = \lambda(\tilde{s}, s)\phi(\tilde{s}). \tag{13}$$

- The mass of new entrants and the probability distribution are stationary

$$\dot{m} = 0 \tag{14}$$

$$\dot{\phi} = 0. \tag{15}$$

3 Characterizing the Equilibrium

In this section, we provide a characterization of the equilibrium and discuss its properties. We compute the equilibrium in two steps. First, we take the value function V and the equilibrium measure of firms ϕ as given and characterize prices p^m, the allocation—that is, the choices of trading partners λ and capital k^m conditional on trading—and the gains from trade V_{trade} that are consistent with market clearing. Second, we solve for (V, ϕ) such that individuals optimize given the menu of prices and terms of trades and ϕ is, in turn, consistent with their decisions.

3.1 Characterizing prices and allocations given (ϕ, V)

As a first step, we show that the equilibrium prices and allocation of capital can be characterized with an *assignment* problem that maximizes the total surplus (as measured using V) by assigning capital subject to preserving the measure ϕ. Define the largest surplus from matching for a pair $(s, \tilde{s})$ as follows:

$$X(s,\tilde{s}) = \max\left\{V(z,k+\tilde{k}), V(s)+V(\tilde{s}), V(\tilde{z},k+\tilde{k})\right\} - \left(V(s)+V(\tilde{s})\right).$$

The three arguments are possible outcomes in a match, namely, type s buys the capital from type $\tilde{s}$, no trade, and type s sells the capital to $\tilde{s}$. We split the measure ϕ into two measures ϕ^a and ϕ^b such that for $s \in \mathcal{S}$ we have

$$\phi^a(s) = \phi^b(s) = \frac{\phi(s)}{2}.$$

For measures $\{\phi^a, \phi^b\}$, an assignment π that maximizes surplus, solves the following maximization problem:

$$Q(\phi, V) = \max_{\pi \geq 0} \int X(s,\tilde{s})\pi(s,\tilde{s}) ds d\tilde{s} \tag{16}$$

such that for $s \in \mathcal{S}$

$$\int \pi\left(s,\tilde{s}\right) d\tilde{s} \leq \phi^a(s) \tag{17}$$

$$\int \pi\left(\tilde{s},s\right) d\tilde{s} \leq \phi^b(s). \tag{18}$$

We label this problem as *P1*. The next theorem shows that we can back out (p^m, k^m, λ) from the solution of *P1* using standard results from the matching literature.[9]

Theorem 1. *Let μ^a and μ^b be the Lagrange multipliers on (17) and (18), respectively, in problem*

[9] See Galichon (2016) for details on the Monge-Kantorovich transportation problem.

P1. Let π be the optimal assignment in problem P1. The functions

$$k^m(s,\tilde{s}) \in \arg\max\{V(z,k+\tilde{k}), V(s)+V(\tilde{s}), V(\tilde{z},k+\tilde{k})\} \tag{19}$$

$$p^m(s,\tilde{s}) = V(z,k^m(s,\tilde{s})) - V(s) - \mu^a(s) \tag{20}$$

$$p^m(\tilde{s},s) = V(z,k^m(\tilde{s},s)) - V(\tilde{s}) - \mu^b(\tilde{s}) \tag{21}$$

$$V_{trade}(s) = \mu^a(s) = \mu^b(s) \tag{22}$$

and measures for all $s,\tilde{s} \subseteq \mathcal{S}$

$$\lambda(s,\tilde{s}) = \frac{\pi(s,\tilde{s}) + \pi(\tilde{s},s)}{\phi(s)} \tag{23}$$

satisfy (2), (3), (6).

Theorem (1) states that the assignment from problem *P1* gives us all the information we need to figure out who trades with whom and at what prices. The intuition is similar to how the welfare theorems operate in frictionless settings. The solution to the planner's problem recovers the allocation and the multipliers on constraints (17) and (18) recover prices.

We can review that intuition in our context. The envelope theorem applied to problem *P1* implies the social gains from having more owners of type s in the market for capital are given by $\mu^a/2+\mu^b/2$. Given the symmetry of the assignment problem, it is easy to show that $\mu^a = \mu^b$ and equals the social gains from trade. Equations (20) and (21) imply that the private gains from trade, which is the change in value from trading net of price paid, are equal to the social gains from trade. This argument connects the equilibrium prices to the multipliers that encode shadow prices for the planner and gives an expression (22) for the gains from trade V_{trade}. The right hand sides of equations (19) and (23) characterizes outcomes of each potential meeting and trading frequencies. These conditions are necessary for private optimality of λ.

3.2 Characterizing (ϕ, V) given $(p^m, k^m, \lambda, V_{trade})$

In the second step, we use the outcomes of the first step to update value functions, V, and the invariant measure, ϕ. The characterization in the first step gives us a handy way of solving the Bellman equation. Given the value of $V_{trade}(s)$ from the solution of problem *P1*, the HJB can be solved using standard methods (for example, finite differences as in Achdou et al. (2021)). The policy functions for investment (θ), trades (λ), and entry or exit (ι) govern the law of motion of the distribution described by equation (10), and for which we find a stationary point (given by condition (15)). Together the two steps characterize the recursive competitive equilibrium as a fixed point. This characterization naturally lends itself to a computational algorithm where we iterate between the steps until convergence.

3.3 Properties of the Equilibrium

The next corollary further sharpens the characterization of the price function p^m.

Corollary 1. *There exists a function $\mathcal{P} : K \rightarrow R_+$ such that*

$$p^m(s, \tilde{s}) = \mathcal{P}(k(s)) \quad \textit{for all } k^m(s, \tilde{s}) < k(s).$$

This corollary says that the pairwise prices only depend on the quantity sold. The intuition for this result is straightforward. The seller's value from trade is equal to the price he extracts from the buyer plus the value of starting anew with zero capital and the current level of productivity. The second component is independent of the trading partner. Thus, conditional on selling to multiple buyers, a seller who maximizes the value from trading must necessarily charge the same price to all buyers. A similar argument from the perspective of the buyer shows that the prices will not depend on the seller's productivity. While our general formulation of pair-wise terms of trade allows for arbitrary gains from matching with any owner of type $\tilde{s}$, our assumption that productivity z is non-transferable delivers an equivalence between our trading protocol for capital and a competitive market with unit demand over differentiated products (indexed by the indivisible size of the capital sold). As such, the prices are only a function of the quantity traded, and summarize this dependence using the function $\mathcal{P}(\cdot)$.[10]

The second property we highlight is that solving for the equilibrium in the market for capital is equivalent to looking for the set of stable matches between owners. A set of matches is stable if there does not exist an alternative price quantity pair $(\hat{k}^m, \hat{p}^m) \neq (k^m, p^m)$, and a pair $(s, \tilde{s}) \in \mathcal{S}^2$

[10] Our model can easily accommodate extensions in which the gains from purchasing capital depend on the productivity of the seller. Among other things, this feature could capture the frequent practice of signing a consulting contract with the seller of a business in order to facilitate the transition to the new ownership.

such that capital allocation $\left\{\hat{k}^m(s,\tilde{s}), \hat{k}^m(\tilde{s},s)\right\}$ satisfies (2)–(3), $\hat{p}^m(s,\tilde{s}) + \hat{p}^m(\tilde{s},s) \geq 0$ and

$$\begin{aligned} V\left(z(s), \hat{k}^m(s,\tilde{s})\right) - \hat{p}^m(s,\tilde{s}) \geq & V\left(z(s), k^m(s,\tilde{s})\right) - p^m(s,\tilde{s}) \\ V\left(z(\tilde{s}), \hat{k}^m(\tilde{s},s)\right) - \hat{p}^m(\tilde{s},s) \geq & V\left(z(\tilde{s}), k^m(\tilde{s},s)\right) - p^m(\tilde{s},s) \end{aligned} \tag{24}$$

with at least one of the last two inequalities being strict.

Corollary 2. *The equilibrium in the market for capital generates a set of stable matches.*

We conclude this section by discussing the efficiency properties of our competitive equilibrium. Given ϕ_0, consider a planner that solves the following maximization problem

$$P(\phi_0) = \max_{\{n_t, b_t, \lambda_t, \theta_t, k_t^m, \iota_t\}} \int_0^\infty \exp(-\rho t) \left[\int \left[y(s, b_t, n_t) - r b_t - c\left(\theta_t(s)\right)\right] \phi_t(s) ds\right] dt$$

such that the (time-dependent analogue of) feasibility conditions (2) and (3), consistency of meeting probability (13), labor market clearing (12), law of motion for the distribution of and the mass of owners (10) and (11) are satisfied.
We label this problem as *P2*. Given linear preferences, maximizing discounted welfare is the same as maximizing discounted net output. We denote a solution to *P2* as stationary if $\phi_t = \phi_0$ for all t. In the next theorem, we show that a stationary recursive equilibrium is efficient.

Theorem 2. *A stationary recursive equilibrium as defined in Definition (1) with the stationary measure ϕ achieves $P(\phi)$ in problem P2. Furthermore, any stationary solution to P2 constitutes a stationary recursive equilibrium.*

The forces towards efficiency were foreshadowed in the formulation of the problem *P1*. Given (ϕ, V), the optimal assignment maximizes output. Beyond the static assignment, there are two additional features in problem *P2*—entry and investment—that need to be addressed. In the appendix, we show that the value of becoming an owner as well as the value of a new unit of capital to the planner coincides with the private value. Thus, the indifference condition for the marginal entrant, and firm optimality with respect to investment, θ, are sufficient to ensure that the allocation is dynamically efficient.

Next, we describe the firm-level data used to parameterize the model.

Table 1: IRS Samples

Business samples	Counts
S corporation population	3,167,266
S corporation sellers	105,162
Sales to S corporations	46,708
to Partnerships	33,462
to C corporations	35,792
Seller-buyer pairs	51,286
S Corporation–S Corporation	28,078
–Partnership	14,040
–C Corporation	9,168

Notes: The 'S corporation population' is drawn from the universe of S corporation filings over the period 1996–2022 and excludes any firms with wage bill under $10,000 or insufficient data for constructing a three-year growth rate in the wage bill. The 'S corporation sellers' is drawn from asset sales recorded on Form 8594 and e-filed with the IRS. By law, both sellers and buyers are required to attach this form to their income tax returns, but the counts reported here avoid double counting if forms from both are e-filed or amended by either party. The 'Sales' counts are based on the number of sales found for the S corporation sellers and are listed by legal form of the buyer. The 'Seller-buyer pairs' are S corporate sellers found on e-filed Form 8594 that have available data on their wage bill in the year prior to the sale and buyer counterparties that have available data on their wage bill in the year after the sale.

4 Data

In this section, we describe firm-level data from U.S. administrative tax records that we will use to calibrate the model. We focus here on corporations that elect Subchapter S status for tax purposes. Under Subchapter S, profits and losses flow through the corporation untaxed and are taxed directly as income to the owner on their individual tax forms. S corporations are now the most prevalent type of corporation in the United States.[11] Unlike other corporate forms, S corporations must have fewer than 100 owners—and typically have only 1 or 2—and the owners must be U.S. citizens or permanent residents. The fact that S corporations are owned by individuals is relevant for our analysis as we are interested in capital transfers between owners of actively-managed businesses. Subchapter C corporations and partnerships do not have the same ownership restrictions: C-

[11] The latest estimates from the Statistics of Income show that S corporations now account for more than three-quarters of all corporate tax filings.

corporate shareholders or partners can be businesses.

Because S corporations are private companies and not directly regulated by the Securities Exchange Commission (SEC), there are no public filing requirements. However, these corporations are required to disclose financial information to tax authorities. Of particular interest here is the reporting of business sales—this information is recorded on IRS Form 8594 (*Asset Acquisition Statement Under Section 1060*) when there is a transfer of business assets that make up a trade or business for either the seller or the buyer.[12] When a business is sold, the IRS must be informed about the allocation of the purchase price across different asset categories. In many cases, the buyer's basis in particular assets is determined only by the amount paid at the time of the sale. For example, values of Section 197 intangible assets such as customer-bases, trade names, and goodwill are typically determined when acquired in a sale. The allocation of price across assets is relevant for the seller who must report capital gains and the buyer who may choose to amortize or depreciate the acquired assets.

In Table 1, we record counts for three IRS data samples. The first sample is the universe of S corporations over the period 1996–2022 with $10,000 or more in wage bill. We also require that the firms in this sample have sufficient data to construct growth rates of the wage bill over a 3-year period—these rates use information for the current tax year and three years prior. We restrict attention to companies with some labor costs to avoid using measures of firm size that are inaccurate if taxpayers underreport incomes or overreport expenses. With these restrictions, we have panel data for a total of 3.2 million S corporations. We refer to this set of firms as the "full sample."

Using the full sample, we construct a second IRS sample of S corporation sellers. These are unique employment identification numbers (or EIN) that can be linked to e-filers of Form 8594 that indicate they have sold a group of assets comprising a business. The database of e-filed Forms 8594 is available over the period 2005-2022. In Table 1, we record counts of sales by legal form of the buyer in which these sellers are the counterparty. The counts of sales can exceed the counts of sellers if an S corporation in our dataset is involved in more than one sale over the sample period.

The final sample in Table 1 is our "trading sample." The trading sample includes pairs of S corporations from the full sample and buyers found on the e-filed Forms 8594, where we restrict

[12] Assets can also be transferred under IRS Section 338 (and recorded on Form 8883), with the buyer purchasing all outstanding shares and assuming all liabilities.

Table 2: Intangible Intensities
U.S. S Corporation Trading Sample

Intangible Intensities	Percentiles 25th	50th	75th
Sales to S corporations	29.2	66.8	87.0
Partnerships	33.3	71.0	90.9
C corporations	39.3	73.4	92.7
All sales	32.1	69.3	89.3

Notes: To ensure that no confidential information is disclosed, reported percentiles are computed as an average of observations around the value listed in the table. Statistics are constructed from the subsample of firms in the trading sample drawn from asset sales recorded on e-filed Forms 8594 and linked to income tax filings. To ensure that no confidential information is disclosed, reported percentiles are computed as an average of observations around the value listed in the table.

attention to those with available data on their business tax filings to construct measures of relative size. More specifically, we construct a sample of seller-buyer pairs for which we have information on the seller's wage bill in the year prior to the sale and the buyer's wage bill in the year after. We restrict attention to sales between employer firms and therefore also include a restriction that the buyer. has a wage bill over $10,000. The trading sample constitutes only a subset of all S corporation asset sales, but offers a rich dataset for those interested in tracking transfers of intangible assets.[13]

Using prices negotiated by the seller-buyer pairs in our trading sample from Table 1, we construct the share allocated to Section 197 intangibles and goodwill (categorized by the IRS as Class VI and VII assets).[14] We call this share the *intangible intensity* and include the following intangible assets: workforce in place; business books and records, operating systems, or any other information base, process, design, pattern, know-how, formula, or similar item; any customer-based intangible; any supplier-based intangible; any license, permit, or other right granted by a government unit; any covenant not to compete entered in connection with the acquisition of an interest in a trade

[13]Transfers that occur between related parties through gifts or bequests are hard—if not impossible—to track. Transfers for which we have no Form 8594 (or Form 8883 in the case of equity sales) should appear for the selling owners as a capital gain on Schedule D, but detailed information on the assets is not available.

[14]When computing the intangible share, we exclude Classes I through III that include cash and marketable securities and include Classes IV through VII that include inventory, fixed assets, real estate, intangibles, and goodwill.

or a business; any franchise, trademark, or trade name; and any goodwill or going concern value. These non-rentable assets are the main focus of our theory.

In Table 2, we report moments of the intangible intensity distribution. Here and below, we compute percentiles as as an average of observations around the value listed in the table.[15] As the data show, intangible assets constitute most of the value transferred in sales of S corporations. The median intangible share of the sale price is 69 if we consider all sales, but hardly different across groups of buyers categorized by legal form. We should note that the remaining assets could well be custom capital and not easily divisible or rentable, making the shares in Table 2 lower bounds for the type of business capital we have in mind. For example, many fixed assets are customized for a business but appear with "fixed assets" (Class V) on Form 8594. If we were to include specialized fixed assets (for example, computers with custom chips, delivery trucks with company logos, and other specialized equipment) along with Section 197 intangibles and goodwill, the share of the price for k in our theory would be even higher.

5 Calibration

In this section, we parameterize the model using moments that can be compared to their IRS counterparts. We start by describing the identification strategy. We then show the main moments regarding firm dynamics, firm size, and business sale patterns that we target in our calibration.

The key parameters include the share of transferable capital in production, α, parameters governing the cost of investment, $c(\theta)$, the distribution of productivity for entrants, $G(z)$, parameters governing the productivity post-entry, (μ_z, σ_z), and the frequency of trade, η. Growth rates by age and dispersion in measures of size for entrants and incumbent firms for the full sample of S corporations are most relevant for estimating productivity distributions and processes. Details from business sales, especially the price relative to wage bill of sellers and the relative wage bills of buyer and seller, are most relevant for estimating the share of transferable capital and the investments costs. Auxiliary data from broker sales are used to estimate the frequency of trades.

[15] This computation ensures that no confidential taxpayer information is disclosed.

Table 3: Model Parameters

Parameter	Expression	Value
Discount rate	ρ	0.05
Entry rate	ψ_{dem}	0.025
Production shares		
Transferable capital share	α	0.13
Rentable capital share	β	0.35
Labor share	γ	0.35
Investment costs		
Cost scale	A	30.0
Cost elasticity	χ	2.0
Depreciation rates		
Transferable capital depreciation	δ_k	0.1
Rentable capital depreciation	δ_b	0.1
Trading rate	η	1.0
Productivity process		
Productivity drift	μ_z	-0.02
Productivity dispersion	σ_z	0.05

Notes: See Sections 2 and 5 for details of these parameters and the choices of functional forms.

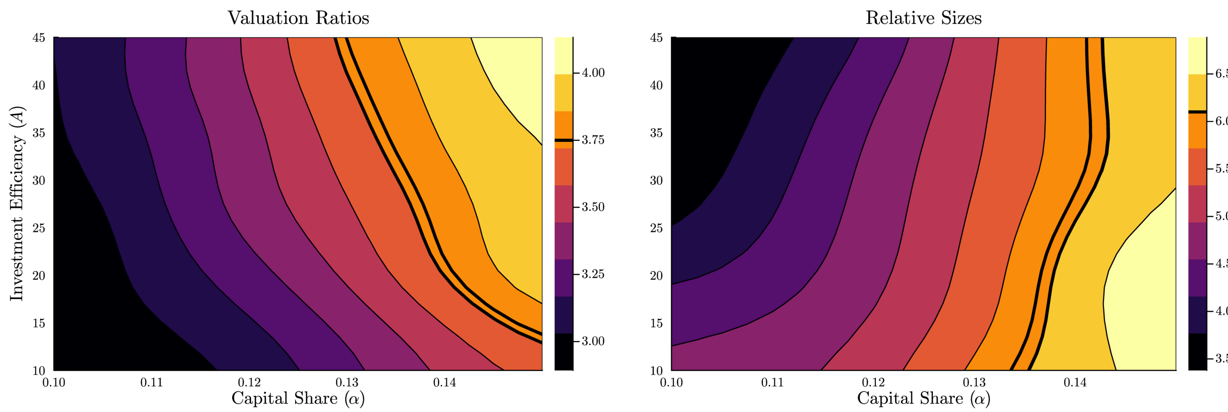
5.1 Identification

The list of parameters is reported in Table 3. The model time period is annual and therefore the discount rate ρ that we use is 5 percent. The entry rate ψ_{dem} is set equal to 1/40 to represent an average working life of 40 years. The parameter ψ governing the option to switch occupations is set to a sufficiently large number to implement our baseline assumption that workers and owners can freely switch anytime.

To parameterize production shares, we first split a measure of value-added for S corporations in the full sample into internal and external payments.[16] Value-added is equal to business receipts less non-labor cost of goods sold. Using publicly available data for S corporations averaged across sample years, we estimate that internal payments to S corporation owners is roughly 30 percent of total value-added. The remaining 70 percent attributed to external payments can then be split

[16]These calculations can be done either with the firm-level data discussed in Section 4 or aggregated data available from the *Statistics of Income (SOI) Tax Stats—S Corporation* statistics at irs.gov.

Figure 1: Identification of Capital Share and Investment Cost



between labor and rentable capital with some allocation of "Other deductions" divided between the two categories. This yields a close to even split in payments and, thus, we set both β and γ equal to 35 percent.

Part of the 30 percent of value-added going to the owner is a payment to transferable capital. How large this payment is depends on the parameter α. To identify this parameter, we use the fact that this share and the cost of investment affect both the value of capital when it is transferred and the relative size of the buyer and the seller. To demonstrate this fact, we first set the investment cost function as follows:

$$c(\theta) = A\frac{\theta^{1+\chi}}{1+\chi}. \tag{25}$$

For a given value of χ, we compute equilibria as we vary values of α and A and, in each case, compute the median valuation ratios and relative sizes of buyer wage bill versus seller wage bill across transactions.

The results are shown in the contour plots of Figure 1. The left panel is meant to capture the price per unit paid by the buyer. We do not observe unit prices in the data but can proxy them with a ratio of the business sale price to some measure of size. In this case, we use the wage bill of the seller as the measure of size and plot the valuation as we vary the two parameters. The contours show that the valuation ratio remains constant as we increase α and decrease A. Buyers are willing to pay more if the factor payments to k are higher, but less if the cost of investing on one's own is lower. The right panel of Figure 1 also shows that the relative size rises with α. If buyers are more productive than sellers, the relative wage bills will be higher the higher is α and the more linear is the production function in k. Holding α fixed, a lower value of A is associated with a higher relative size as it is less costly for the buyer to scale up when investment costs are lower. When we overlay the two panels of Figure 1, we have a way of inferring the (α, A) pair consistent with observed valuation ratios and relative sizes.

The moments in Figure 1 are affected by the investment cost elasticity χ, which can be identified by dispersion in the valuation rations and relative sizes. We set $\chi = 2$ in our baseline calibration. Similarly, the rates δ_k and δ_b are not critical for the comparison statistics. These can be set to values used in Bhandari and McGrattan (2021), who rely on data from the General Accounting Office and the Bureau of Economic Analysis to estimate these rates.

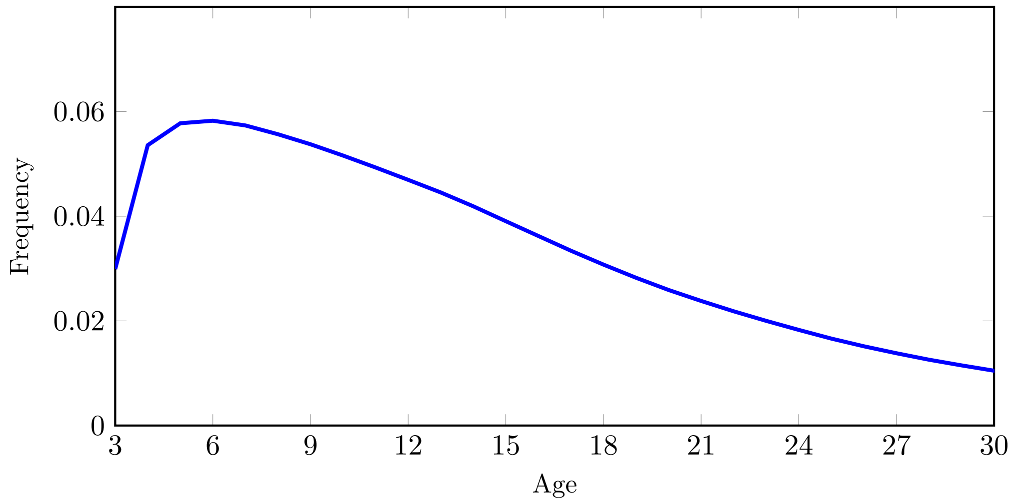
The entry rate, η, does affect firm dynamics, especially the standard deviation of the marginal products of capital. In the baseline, we assume that a trading opportunity arises once a year. Here, we rely on broker data from Pratt's Stats (now DealStats) that show close to one year spells between hiring the broker and the business sale. (See Bhandari and McGrattan (2021) for details on the broker data.)

The final set of parameters are related to the process for productivity, μ_z and σ_z. As it is standard in models of firm dynamics, the moments that motivate our choices are the age distribution of firms, the annualized 3-year growth rate of firm wage bills, and the distribution of the log wage bills. In addition to these parameters, we calibrate the initial distribution $G(z)$ by matching the distribution of the log wage bills across young firms and the fraction of individuals who become business owners.

5.2 Empirical Moments

Using dates of business establishment, we compute a business age for each S corporation in our full IRS sample. In Figure 2, we plot the age distribution for these corporations, which starts at age three given our sample construction. With delays in hiring and entrants starting mid-year, the firm counts grow initially, peaking by age six. In Figure 3, we plot the distribution of annualized 3-year growth rates for full IRS sample, which are constructed using the wage bills. Here, we plot the 25^{th}, 50^{th}, and 75^{th} percentiles at each age. As the figure shows, the median growth is 25 initially and falls to 6 the following year.

Figure 2: Age Distribution
Sample of U.S. S Corporations

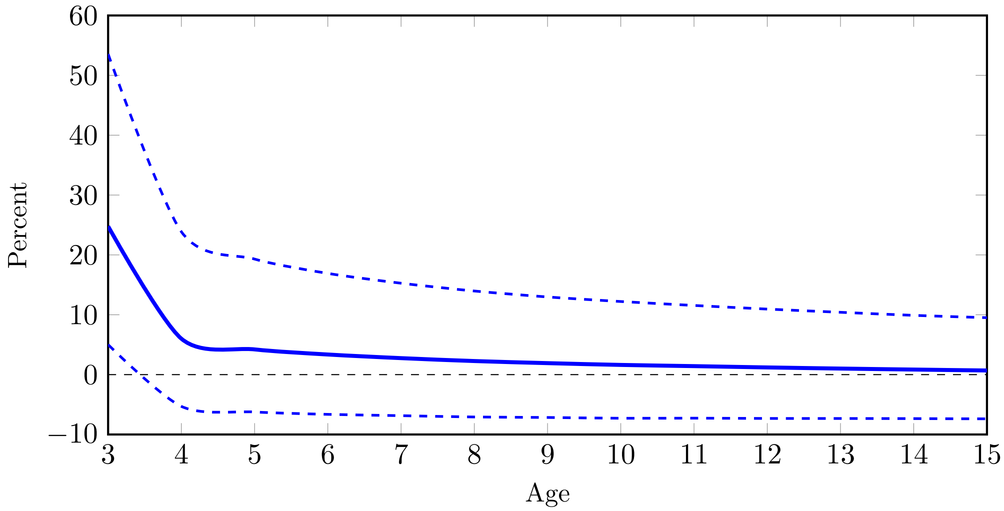


Notes: The age distribution is constructed using the universe of S corporation filings over the period 1996–2022 and excludes any with wage bill under $10,000 or insufficient data for constructing a three-year growth rate in the wage bill.

In Table 4, we report summary statistics for our full sample. The first rows are population moments for the data plotted in Figures 2 and 3. The interquartile range of business ages for this sample is 8 to 21. The interquartile wage growth is -7 to 12. We also report these percentiles for the log of the wage bill, in the case of entrants—age-3 firms in full IRS sample. Comparing the interquartile ranges, we do not find large differences between the statistics for the age-3 firms and those for all ages of the population. We see in Figure 2 that the growth is rapid in the few years following the establishment of the business, and thus we should not be surprised to find small differences between those that are age 3 and the remaining population.

In Table 5, we report key statistics from our IRS trading sample. In the top panel of the table, we report valuation ratios, which are computed as the ratio of the total price paid for a group of business assets divided by the wage bill of the seller in the year before the sale. We report sales to S corporations, C corporations, and partnerships separately since C corporations and partnerships owned by other business entities tend to be larger in size. Roughly half of the counterparties in sales involving S corporations are S corporations themselves. For the median sales across the three categories of exchange, we find a range of valuation ratios of 2.4 to 4 times the seller's wage bill.

Figure 3: Distribution of Annualized 3-Year Growth by Age
Sample of U.S. S Corporations



Notes: The growth distribution is constructed using the universe of S corporation filings over the period 1996–2022 and excludes any with wage bill under $10,000 or insufficient data for constructing a three-year growth rate in the wage bill.

Table 4: Summary Statistics
Sample of U.S. S Corporations

Statistic	Percentiles 25th	50th	75th
Business Age	8.0	13.0	21.0
Wage Growth	−6.8	1.4	11.6
Log Wage Bill: Entrants	11.0	11.7	12.5
Population	11.1	11.9	12.8

Notes: These statistics are based on the universe of S corporation filings over the period 1996–2022 that excludes any firm with wage bill under $10,000 or insufficient data for constructing a three-year growth rate in the wage bill. To ensure that no confidential information is disclosed, reported percentiles are computed as an average of observations around the value listed in the table.

When parameterizing the model, we aim for valuation ratios in this range.

In the bottom panel of Table 5, we report statistics for a measure of relative size: the ratio of

Table 5: Summary Statistics
Sample of U.S. S Corporation Sellers

Statistic	Percentiles 25th	50th	75th
Valuation Ratios			
Sales to S corporations	1.0	2.4	5.2
Partnerships	1.4	3.5	8.6
C corporations	1.5	4.0	9.9
All sales	1.2	2.9	6.7
Relative Wage Bill Sizes			
Sales to S corporations	0.7	1.4	5.6
Partnerships	1.0	2.8	17.4
C corporations	2.2	14.9	130.7
All sales	0.9	2.1	13.5

Notes: Statistics are constructed from the subsample of firms drawn from asset sales recorded on e-filed Forms 8594. The "valuation ratio" is the ratio of firm sale price to seller's wage bill in the year prior to the sale. The "relative wage bill size" is the ratio of the buyer's wage bill in the year after the sale to the seller's wage bill in the year prior to the sale. To ensure that no confidential information is disclosed, reported percentiles are computed as an average of observations around the value listed in the table.

the wage bill of the buyer a year after the sale to the wage bill of the seller a year before the sale. As compared to valuation ratios, we find much more heterogeneity in relative sizes. For our sample of S corporations, the median size ratios across the different legal form categories range from 1.4 where the buyers are S corporations to 15 where the buyers are larger C corporations.

Because there is a wide range of values for the relative size of wage bill by legal form, we also report information on this key statistic by the size of the S corporation seller. More specifically, we take the log of the wage bills for all sellers in the IRS trading sample and assign them to 10 bins: below 11, 8 equally-spaced bins between 11 and 15, and above 15. For each bin, we compute the median wage bill of the seller and the interquartile wage bills of the buyers who are counterparties to the sales reported on Form 8594. In Figure 4, we plot these results on a log scale. The lower line of the figure is the 25^{th} percentile for the buyer wage bills, which is very close to the 45 degree line. This means that most buyers are larger in size than the sellers. The 75^{th} percentile is about

Figure 4: Buyer and Seller Wage Bills By Seller Size

Notes: The sellers are S corporations in the IRS trading sample. Their wage bills one year prior to the sale of the business are assigned to 10 bins and medians of each bin are used for the figure's x-axis. The y-axis is the interquartile range of wage bills in the year after the sale for buyers who were counterparties on Form 8594 to sellers assigned to the bin. To ensure that no confidential information is disclosed, reported percentiles are computed as an average of observations around the value listed in the table.

one order higher—or 10 times. A clear pattern emerges in that the ratio is nearly constant across the seller's size distribution.

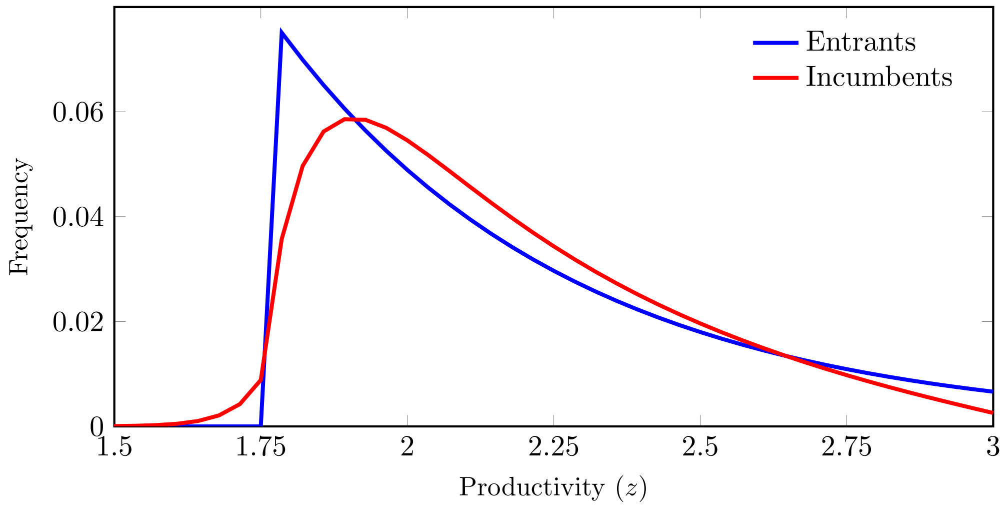
5.3 Model Moments

Starting with the distribution of entrant productivity, $G(z)$, shown in blue in Figure 5, we compute the equilibrium distribution, $\phi(z)$, shown in red. Using this distribution as sample weights, we simulate monthly data for a long panel of firms and construct annual statistics comparable to those obtained from the data.

Using the model data, we first compute an age distribution of firms shown in Figure 6. As with the U.S. data, we require that the model firms have data for at least 3 years in the sample. Because these firms might start in a month part way through the tax year, there is an initial rise in the distribution as in the data. Comparing Figures 2 and 6, we find significant dispersion in both the observed and predicted age distribution.

In Figure 7, we plot the annualized 3-year growth of firm wage bills. The solid line is the median

Figure 5: Distributions for Entrant and Incumbent Productivities
Sample of Model Corporations



Notes: The distribution for entrants is an input in the calibration, namely, $G(z)$, and the distribution of incumbents is the equilibrium output, $\phi(z)$. See Sections 2 and 5 for details.

growth and the dashed lines are the 25$^{\text{th}}$ and 75$^{\text{th}}$ percentiles at each age. As in the case of the U.S. data shown in Figure 3, we find a large drop in the growth rate at the start and a steady interquartile range between ages 5 and 15. In Figure 8, we plot the predicted distribution of log wages for the model sample of firms. The ratio of the 75$^{\text{th}}$ to 25$^{\text{th}}$ percentiles is 10, which is in line with the data.

In Table 6, we report the valuation ratios and relative sizes for the model. As we noted earlier, there are differences in these estimates if we condition on the legal form of the counterparty. Thus, we parameterize the model with a range of estimates in mind. In the case of the valuation ratio—which is the ratio of the transferred business sale price to the seller's wage bill—we find a median value of 3.7 for the model, within the range of estimates we found in the data. In the case of the relative wage bills for the buyers and sellers, in the model we find a similarly wide interquartile range as in the data, although the median value of 6 is higher than its empirical counterpart if buyers across all legal forms are included.

Table 6: Predicted Valuation Moments

Statistic	Percentiles 25th	50th	75th
Valuation Ratios	3.1	3.7	4.8
Relative Wage Bill Sizes	2.3	6.0	18.5

Notes: The valuation ratio is the sale price $\mathcal{P}(k(s))$ divided by the seller's wage bill $wn(s)$. The relative size is ratio of the buyer's to seller's wage bill.

6 Model Predictions

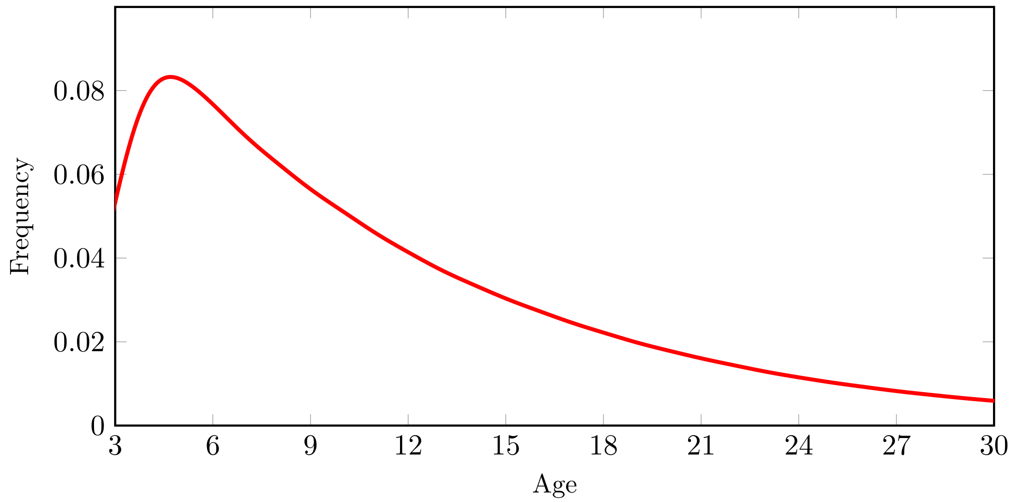
In this section, we report on key model predictions that have no counterpart in U.S. data but are relevant for policy analysis. The first is the dispersion in the marginal product of capital, which is a central statistic in the literature on capital misallocation. We compute this dispersion in our baseline and show how it changes as we vary the frequency of trade. The second set of statistics relate to private business wealth, on which information is typically available only when transferred, but central in the literature on wealth inequality. We report the model's predictions for income yields, the share of value that is transferable, and estimates of total private value to output in private business.

6.1 Dispersion in Marginal Product of Capital

Because building businesses takes time, whether owners invest in or purchase capital, the marginal products of capital, $\alpha y(s)/k(s)$, are not equated across firms. The red line in Figure 9, plots the distribution of marginal products in our baseline model, which shows clearly that there is significant heterogeneity across firms. In this case, the distribution has a standard deviation in logs of 57 percent.

In the baseline sample, we choose $\eta = 1$ to replicate a trading frequency of once per year. To see how the dispersion changes as we vary this statistic, we consider two alternatives: a trading frequency of once per month ($\eta = 12$) and no trade at all ($\eta = 0$). The results are shown alongside the baseline in Figure 9. Here we see that there is significant dispersion in all three cases. Not surprisingly, without trade, the standard deviation in logs is higher—roughly 90 percent. Somewhat

Figure 6: Age Distribution
Sample of Model Corporations



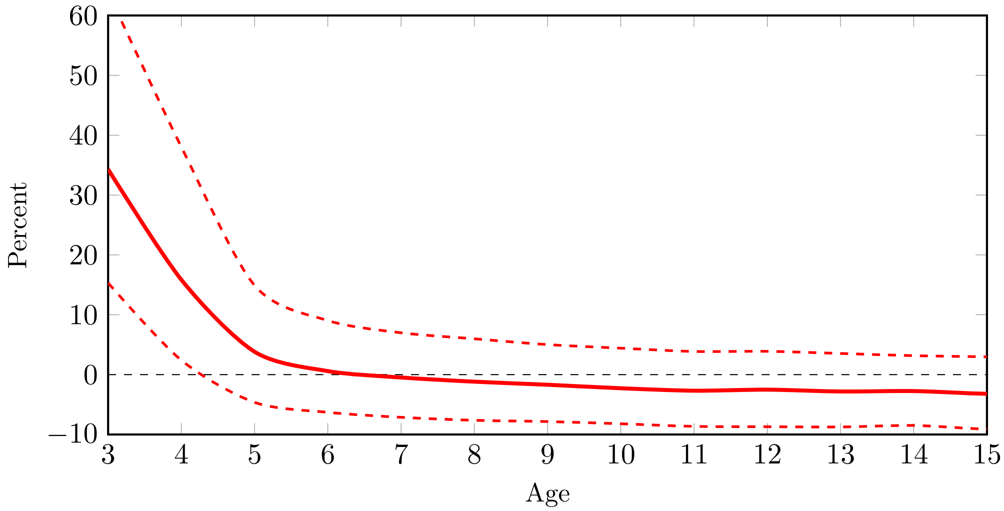
Notes: The age distribution is constructed using the universe of model corporations with sufficient data to construct a three-year growth rate in wage bill.

more surprising is that with very frequent trade, in this case monthly, the standard deviation is still high—roughly 30 percent.

What these results tell us is that the indivisibility of trade is playing an important role quantitatively. Even if we assume that owners could speed up the transactions, say by paying higher costs to brokers, the fact that business assets are sold as a group and to buyers interested in a particular size of sale implies that the reallocation of capital will necessarily be gradual (except in the empirically uninteresting idealized limit).

Our findings on the measured dispersion in the marginal product of capital can be compared with studies in the literature that use microdata on utilization of plant and equipment across production units. Notably related are studies such as Cooper and Haltiwanger (2006), Hsieh and Klenow (2009), and Asker et al. (2014), which use plant-level data from the Annual Survey of Manufactures (ASM), as well as David and Venkateswaran (2019) and David et al. (2022), which utilize accounting data to assess the dispersion in the marginal product of capital for publicly-traded U.S. firms. While these studies differ in time frames, firm samples, and types of capital, their estimates of the standard deviation of the log of the marginal product of capital generally range

Figure 7: Distribution of Annualized 3-Year Growth by Age
Sample of Model Corporations



Notes: The growth distribution is constructed using the universe of model corporations with sufficient data to construct a three-year growth rate in wage bill.

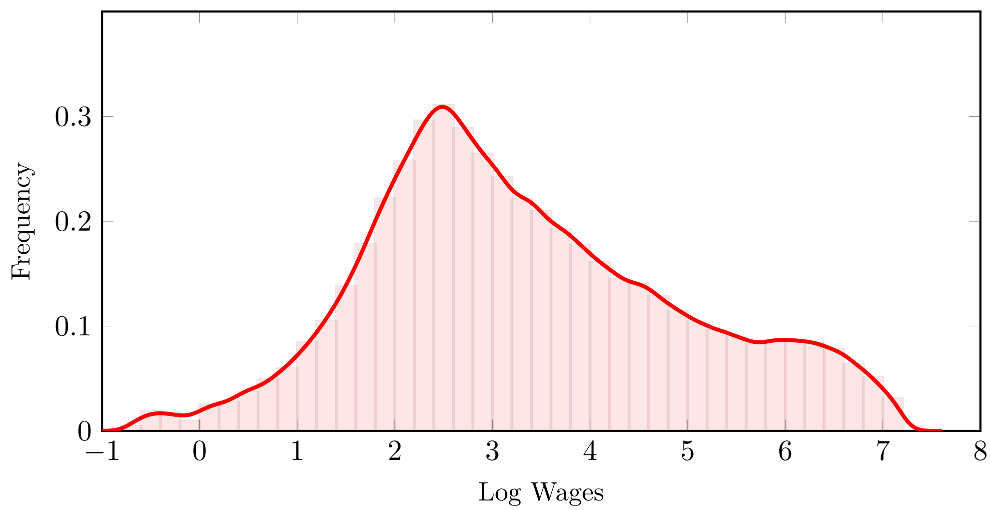
between 60 and 100 percent. Our theory-guided estimates in the case of non-rentable, indivisible capital in private businesses fall near the lower end of this range for our baseline calibration.

6.2 Business Wealth

The model we work with has two concepts of business wealth. The first is the present discounted value of owner dividends, $V(s)$, which captures returns to both transferable capital k and non-transferable capital z. More familiarly, this value can be interpreted as the private-business counterpart of a stock price for shares of a publicly-traded corporation. We use these values to estimate variation in business returns. The second measure of business wealth is often reported in surveys of consumer finances that ask respondents to estimate the price of the business if it were sold today. This measure in our model is the price of transferable capital, $\mathcal{P}(k(s))$. We use these values to estimate variation in transferable shares of private business wealth and later as inputs when comparing the effects of taxing businesses.

In Table 7, we report distributional statistics for income yields and transferable shares. The *income yield* is a common measure of the return to business and is given by the ratio of owner

Figure 8: Distribution of Log Wages
Sample of Model Corporations



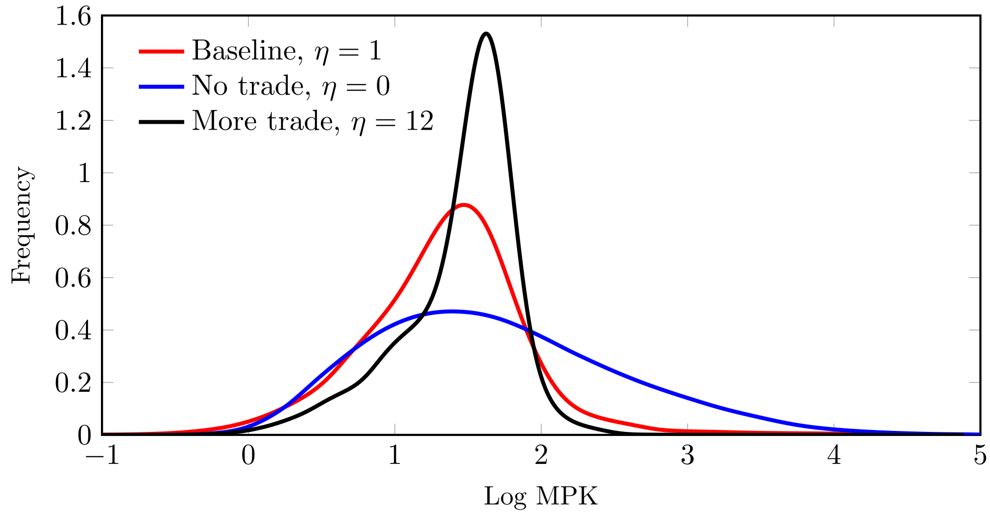
Notes: The distribution of log wages is constructed using the universe of model corporations with sufficient data to construct a three-year growth rate in wage bill.

income $(1 - \beta - \gamma)y(s) - c(\theta(s))$ to business value $V(s)$. The predicted returns again highlight the heterogeneity across business outcomes, with estimates ranging from 2.6 percent at the 5^{th} percentile of the distribution to 17.4 percent at the 95^{th}. The average return is 8.7 percent, which is in line with pre-tax stock returns for U.S. publicly-traded companies. We also report the aggregate return, which is computed as the ratio of total owner net income to total value. Our estimate of 13.6 percent can be compared to economy-wide estimates from national accounts and flow of funds.

The third column of Table 7 shows the *transferable share*, which is the ratio of transferable capital to the total value. As with income yields, we find significant heterogeneity in shares. At the 5^{th} percentile, the value of transferable capital is equal to 9.8 percent times the total business value, and at the 95^{th} percentile, the ratio is 43.6 percent. If we compute the equal- or value-weighted shares, we find 23 percent and 27 percent, respectively. If we were to compute ratios of wealth to total output in the model economy, we would find that the total transferable capital, $\int P(k(s))\,ds$, is 0.60 times total output. It follows from Table 7 that the total private business wealth, $\int V(s)\,ds$, is estimated to be 2.21 times total output in private business.

We turn next to evaluating different forms of business taxation, comparing in particular impacts

Figure 9: Predicted Distributions of Log MPK
Varying Trading Rate η



Notes: MPK is the marginal product of capital given by $\alpha y(s)/k(s)$ and η is the parameter governing trading frequency. The distributions of log MPK are constructed using the universe of model corporations with sufficient data to construct a three-year growth rate in wage bill.

on entry, investment, trading, and welfare.

7 Tax Policy Analysis

There is an active public debate on how to tax businesses, particularly regarding whether to tax business income, wealth, or capital gains. In standard models with perfect financial markets, the classical Atkinson and Stiglitz (1976) result on uniform commodity taxation implies a zero tax rate on the capital value or the returns to capital. Taxing capital or returns would introduce an intertemporal wedge, effectively taxing consumption at different times at varying rates. In the context of business taxation, this insight supports proposals that advocate taxing only distributions (business income minus the cost of investment) as a way to raise revenue without creating distortions.

In our framework, applying the Atkinson-Stiglitz result would imply allowing deductions for both entry costs and investment costs. However, this approach is impracticable in our setting because the entry cost and investment costs depend on the opportunity cost of the owner's time.

Table 7: PREDICTED INCOME YIELDS AND TRANSFERABLE SHARES

STATISTIC	INCOME YIELD	TRANSFERABLE SHARE
Percentiles		
5^{th}	2.6	9.8
10^{th}	3.1	11.0
25^{th}	4.6	12.3
50^{th}	7.5	21.3
75^{th}	12.7	30.3
90^{th}	16.5	38.1
95^{th}	17.4	43.6
Average	8.7	23.1
Aggregate	13.6	27.0

Notes: The income yield is the ratio of owner income, $y(s) - wn(s) - rb(s) - c(\theta(s))$, to business value $V(s)$. The transferable share is the ratio of the transferable value $\mathcal{P}(k(s))$ to the total value $V(s)$. Statistics related to the distribution are are reported as well as the ratios of economy-wide aggregates.

This motivates us to explore alternative, second-best approaches of taxing businesses.

When financial markets are imperfect, Guvenen et al. (2023) highlight the distinction between taxing financial wealth and the return on financial wealth in environments where owners differ persistently in their business management abilities. As discussed in Section 6, our model generates persistent heterogeneous returns on capital, allowing us to revisit the insights of Guvenen et al. (2023) regarding the effects of taxing capital values versus capital returns.

Because there has been less theoretical development, the implications of taxing capital gains from a transfer of capital between owners are not as well understood. Much of the existing public finance literature is empirical and focused on measuring the elasticity of a broader notion of capital in response to changes in capital gains tax rates. As we discussed in Section 1.1, Chari et al. (2003) provides one of the few theoretical assessments of the impact of taxing business transfers within the Holmes and Schmitz (1990) model. We revisit their calculations in our setting and also evaluate the elasticities estimated in the empirical literature.

Table 8: Predicted Tax Policy Changes

Statistic	Business Income	Capital Value	Capital Gains
Wage	−0.5	−1.3	−6.0
Mass of firms	0.8	−7.1	−32.3
Fraction traded	3.8	−5.8	−64.6
Average investment	0.3	−0.9	−1.6
Dispersion in MPK	3.5	−2.5	2.2

Notes: The mass of firms is the equilibrium m. The fraction traded is the amount of capital $k(s)$ transferred in the period relative to economy-wide capital. The average investment is the average value for $\theta(s)$. The dispersion in marginal product of capital (MPK) is the standard deviation of the log of the marginal product of capital, $\alpha y(s)/k(s)$. The wage is the equilibrium w.

7.1 Comparing Business Taxes

As discussed above, our model provides a useful setting to frame the discussion of how to tax businesses because it has predictions for flows, stocks, transfers, and valuations of business capital. In this section, we study various forms of business taxation in the model economy.

We consider the problem of a government that wants to raise a certain amount of revenues using a linear tax on either owner income, $y(s) - wn(s) - rb(s)$, or some measure of business wealth. In the case of wealth, we consider taxing the value of the transferable capital each year—assuming the government is able to assess the value $\mathcal{P}(k(s))$ of the assets—or taxing the realized gains after the business sells.[17] We proxy the welfare effect of the policy with steady-state wages, which are proportional to the value of being a worker.

We compare aggregate outcomes for three taxed economies relative to a no-tax baseline. In all cases, we raise revenue equal to 1.5 percent of output in the baseline economy. The tax rates on income, capital, and capital gains needed to raise this sum are 4.8 percent, 2.7 percent, and 50 percent, respectively. The impacts of interest are the changes in firm entry, the changes in the amount of capital traded, the own investment in the business, the dispersion in marginal products of capital, and welfare.

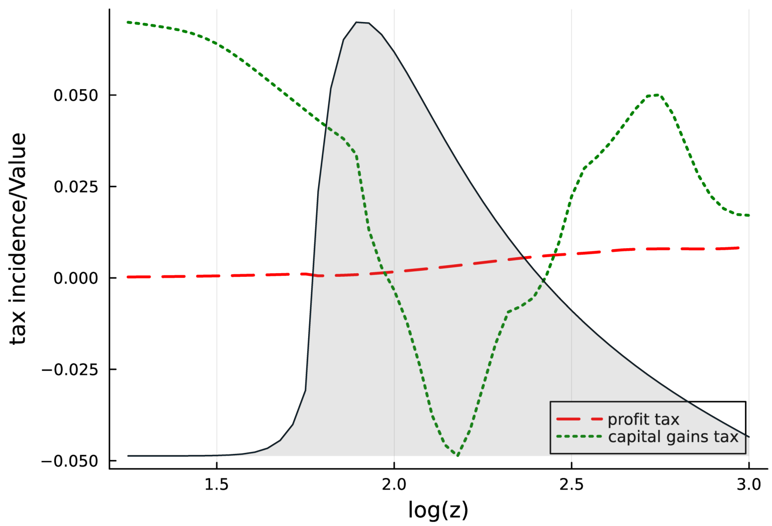
[17]For example, as in the case of housing, we assume that the IRS could use financial data and valuations ("comps") from recent sales of businesses that are a similar size and in the same industry.

Results are shown in Table 8. Our main finding is that taxing business income is much less distortive than taxing either capital values or capital gains. In the first row of Table 8, we report the impact on wages, which is our measure of welfare. The bottom line is the stark difference between welfare losses with an increase in the income tax rate compared to the capital gain tax rate: -0.5 percent versus -6 percent and the tax on assessed value of capital falls in between. This concisely summarizes the distortive nature of taxing capital gains in a rich model environment with entry, investment, and trading decisions on the part of business owners.

All taxes distort entry, investment, and capital reallocation, but they differ in whom they primarily affect. Starting with impacts on entry shown in the second row of Table 8, we see that a tax on owner net income has almost no effect on the mass of firms while the tax on capital gains triggers a collapse in entry equal to -32.3 percent. The tax on business income does not significantly deter entry, as the high-productivity owners affected by this tax are inelastic and likely to enter regardless. Next, consider how taxes distort the fraction of capital traded and investment. These results are shown in the third and fourth row of Table 8. The decline in the fraction traded is -64.6 percent in the case of the capital gains tax. This decline is driven by a "lock-in" effect, with capital remaining with less productive owners due to discouraged trade. The tax on business income, however, is less disruptive to trade and thus less distortive overall. If we consider the effects of the capital gain tax on investment, we find that they are small due to an offsetting general equilibrium force, namely, that productive owners substitute toward investment when prices of capital rise. If we average across all owners, investment does fall but not by much.

In terms of distortions, a tax on capital assessed annually falls between the tax on business income and the tax on capital gains. It is broad-based, similar to the tax on business income, but its incidence mainly falls on high-capital, medium-productivity owners. Like the capital gains tax, a tax on the assessed value of capital discourages entry and investment compared to a tax on business income. However, in one respect the capital tax is different, specifically in its impact on dispersion in marginal products of capital. In the fifth row of Table 8, we report changes in dispersion for the three tax policies and find that the dispersion is lower with the annual tax on capital. Because this tax effectively redistributes resources from relatively high-capital, low-productivity owners to relatively low-capital, high-productivity ones, we find a decline in dispersion of marginal products of capital. Quantitatively, when considering overall welfare, this reduction in measured misallocation

Figure 10: Tax Incidence



Notes: The red dashed line is the average income tax payment as a fraction of firm value for each productivity level. The average is taken with respect to the conditional capital ownership distribution in the baseline economy. The green dotted line is the average loss in revenues (additional price) from selling (buying) capital as a fraction of firm value. The averages are computed using the distribution of trades (shown in gray) in which a firm with a given productivity is involved in the baseline economy.

is modest and outweighed by the resulting distortions in entry and investment.

In Figure 10, we summarize the tax incidence for a tax on business income and a tax on capital gains. The red dashed line shows the average income tax payment as a fraction of firm value for each productivity level, where the average is taken with respect to the conditional capital ownership distribution in the baseline economy. The green dotted line in Figure 10 shows the economic incidence of the capital gain tax. It is computed as the average loss in revenues from selling capital—or additional price to be paid when buying capital—as a fraction of firm value, with averages computed using the distribution of trades in which a firm with a given productivity is involved in the baseline economy. As the figure clearly shows, the incidence of the tax on business income mainly falls on high-productivity owners who earn the most, whereas the tax on capital gains impacts trading firms, which are at both ends of the productivity spectrum.

Given the distortive effects of capital gains taxes, it is natural to ask whether the relevant elasticities are high in our setting. While direct data on the elasticity of business transfers in response to changes in capital gains taxes is unavailable, we can compare our model-implied elasticity estimates to empirical findings on the responsiveness of the tax base of all capital gains to changes in capital gains tax rates. Using state-level variation in capital gains tax rates, Gentry and Bakija (2014) and Agersnap and Zidar (2021) estimate these elasticities to be in the range of -0.3 to -0.66. The Joint Committee on Taxation (JCT) and the U.S. Treasury use higher elasticity estimates of -0.7 and -1, respectively. Our baseline calibration implies an elasticity of -0.6, which aligns with the empirical range found in the literature.

8 Conclusion

Theory has been developed to study the reallocation of capital through business sales. The capital we modeled is neither divisible nor typically sold in centralized markets, but constitutes most capital transferred in private business sales in the United States. We used the theory to study firm dynamics, business wealth, and business taxation. With parameters disciplined by administrative tax data, the theory predicts significant dispersion in marginal products of capital, returns to business wealth, and heterogeneity of transferable capital shares. Comparisons of taxes on business incomes versus different measures of wealth reveal a clear welfare ranking of income taxes over wealth taxes.

In order to keep the mathematics and numerics as transparent as possible, we made certain assumptions that can be relaxed in future work. We used quasi-linear preferences to exploit tools from the matching literature and prove efficiency. We assumed capital is indivisible but otherwise homogeneous for tractability. These choices, among others, must ultimately be disciplined by additional observations from the data.

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Appendix

A Proof of Theorem 1

We prove the theorem by using duality to cast the Monge-Kantorovich problem in a form that highlights the properties of the allocation, in particular the feasibility of capital and prices and the stability of the equilibrium. Consider problem $P1$,

$$\max_{\pi\geq 0,\pi_o^a\geq 0,\pi_o^b\geq 0} \quad \Sigma_{s,\tilde{s}} X\left(s,\tilde{s}\right)\pi\left(s,\tilde{s}\right)+\Sigma_s V\left(s\right)\pi_o^a\left(s\right)+\Sigma_{\tilde{s}} V\left(\tilde{s}\right)\pi_o^b\left(\tilde{s}\right)$$

$$\begin{aligned} s.t. \quad & \Sigma_{\tilde{s}}\pi\left(s,\tilde{s}\right)+\pi_o^a\left(s\right)=\phi\left(s\right)/2 \\ & \Sigma_s\pi\left(s,\tilde{s}\right)+\pi_o^b\left(\tilde{s}\right)=\phi\left(\tilde{s}\right)/2 \end{aligned}$$

Formulate the Lagrangian,

$$\begin{aligned} & \max_{\pi\geq 0,\pi_o^a\geq 0,\pi_o^b\geq 0} \Sigma_{s,\tilde{s}} X\left(s,\tilde{s}\right)\pi\left(s,\tilde{s}\right)+\Sigma_s V\left(s\right)\pi_o^a\left(s\right)+\Sigma_{\tilde{s}} V\left(\tilde{s}\right)\pi_o^b\left(\tilde{s}\right) \\ + \min_{\mu^a,\mu^b} & \Sigma_s\mu^a\left(s\right)\left[\phi\left(s\right)/2-\Sigma_{\tilde{s}}\pi\left(s,\tilde{s}\right)-\pi_o^a\left(s\right)\right]+\Sigma_{\tilde{s}}\mu^b\left(\tilde{s}\right)\left[\phi\left(\tilde{s}\right)/2-\Sigma_s\pi\left(s,\tilde{s}\right)-\pi_o^b\left(\tilde{s}\right)\right] \end{aligned}$$

Using the minimax theorem,

$$\begin{aligned} & \min_{\mu^a,\mu^b} \quad \Sigma_s\mu^a\left(s\right)\phi\left(s\right)/2+\Sigma_{\tilde{s}}\mu^b\left(\tilde{s}\right)\phi\left(\tilde{s}\right)/2 \\ & +\max_{\pi\geq 0} \quad \Sigma_{s,\tilde{s}}\left[X\left(s,\tilde{s}\right)-\mu^a\left(s\right)-\mu^b\left(\tilde{s}\right)\right]\pi\left(s,\tilde{s}\right) \\ + & \max_{\pi^a\geq 0,\pi^b\geq 0} \Sigma_s\pi_o^a\left(s\right)\left[V\left(s\right)-\mu^a\left(s\right)\right]+\Sigma_{\tilde{s}}\pi_o^b\left(\tilde{s}\right)\left[V\left(\tilde{s}\right)-\mu^b\left(\tilde{s}\right)\right] \end{aligned}$$

or equivalently,

$$\min_{\mu^a,\mu^b} \Sigma_s \mu^a(s)\,\phi(s)/2 + \Sigma_{\tilde{s}} \mu^b(\tilde{s})\,\phi(\tilde{s})/2$$
$$s.t. \mu^a(s) + \mu^b(\tilde{s}) \geq X(s,\tilde{s})$$
$$\mu^a(s) \geq V(s)$$
$$\mu^b(\tilde{s}) \geq V(\tilde{s})$$

We denote the latter problem the dual of $P1$. Observe that the dual problem is invariant to swapping the labels a and b on μ, which implies that at an optimal solution $\mu^a = \mu^b$. We conclude the proof of the theorem in two steps.

First, it is easy to see that conditional on a match, the choice of capital is feasible by definition of X. Hence conditions (1) and (2) are satisfied. In addition, for matches that are formed in equilibrium, that is, $\pi(s,\tilde{s}) > 0$, $\mu^a(s) + \mu^b(\tilde{s}) = X(s,\tilde{s})$. This result follows from complementary slackness of the dual problem. Summing up (17) and (18) guarantees that (3) is satisfied (with equality). It also immediately follows that (19) and (20) satisfy the restrictions (9a) and (9b) on the measures.

Second, we show that the pair (p^m, k^m) satisfies pairwise stability given V. Suppose, by contradiction, that it is not the case. That is, there exists a pair $(s,\tilde{s})$, feasible capital allocation $k^m(s,\tilde{s})$ and prices $\hat{p}$, such that

$$V\left(z, \hat{k}^m(s,\tilde{s})\right) - \hat{p}(s,\tilde{s}) \geq \mu(s)$$
$$V\left(\tilde{z}, \hat{k}^m(s,\tilde{s})\right) - \hat{p}(\tilde{s},s) \geq \mu(\tilde{s})$$

with at least one inequality being strict, and

$$\hat{p}(s,\tilde{s}) + \hat{p}(\tilde{s},s) \geq 0.$$

Without loss of generality, we consider the capital allocation that would maximize the sum of the values of the deviating pair. Summing up the values from deviating we get

$$X(s,\tilde{s}) - (\hat{p}(s,\tilde{s}) + \hat{p}(\tilde{s},s)) > \mu(s) + \mu(\tilde{s}).$$

Using the first constraint in the dual problem,

$$\mu(s) + \mu(\tilde{s}) \geq X(s,\tilde{s})$$

which implies $\hat{p}(s,\tilde{s}) + \hat{p}(\tilde{s},s) < 0$, a contradiction that concludes the proof.

B Proof of Theorem 2

We prove efficiency under an assumption that productivity space is discrete. This implies a discrete space of types, $\mathcal{S} = \{s_1, ..., s_N\}$. We do so to keep notation simple, but the result extends naturally to a continuum of types. Accordingly, let $g(s)$ be the probability mass function of entrants of type s. Given ϕ_0, consider a planner that solves the following maximization problem.

$$P_t(\phi_0) = \max_{\{\lambda_t, \lambda_{o,t}, \theta_t, k_t^m, n, m_t\}} \int_t^\infty \exp(-r(\tau - t)) \{\Sigma_{s\in S} [y(s,n) - c(\theta_t(s))] \phi_t(s) - n_0 m_t\} \, \mathrm{d}\tau$$

subject to

$$\dot{\phi}_t(s) = \Gamma(s, \phi_t; \lambda_t, \lambda_{o,t}, \theta_t, \phi_e, k_t^m) \quad \forall s \in \mathcal{S},$$

feasibility of k^m and $\phi_e(s,m) = mg(s)$ for all $s \in \mathcal{S}$ and

$$N_t^w = \int n_t(s)\phi_t(s) + n_0 m_t$$

$$\dot{N}_t^w = \delta(N - N_t^w) - m_t$$

Set-up

The recursive formulation of the planner's problem is

$$
\begin{aligned}
rP\left(\phi_t, N_t^w\right) = & \max_{\{\lambda_t,\lambda_{o,t},\theta_t,k_t^m,n,m_t\}} \quad \Sigma_s \left[y\left(s,n\right) - c\left(\theta_t\left(s\right)\right)\right]\phi_t\left(s\right) + \sum_{\hat{s}} \frac{\partial P\left(\phi_t\right)}{\partial \phi_t(\hat{s})} \Gamma\left(\hat{s},\phi_t;\lambda_t,\lambda_{o,t},\theta_t,k_t^m,m_t\right) \\
& + \frac{\partial P\left(\phi_t, N_t^w\right)}{\partial N_t^w} \delta(N - m_t - N_t^w).
\end{aligned}
$$

In what follows, we omit some arguments of the functions Γ for brevity. The optimality conditions are

$$
\begin{aligned}
c'\left(\theta_t\left(s\right)\right)\phi_t\left(s\right) &= \Sigma_{\hat{s}} \frac{\partial P\left(\phi_t\right)}{\partial \phi_t\left(\hat{s}\right)} \frac{\partial \Gamma\left(\hat{s}\right)}{\partial \theta_t\left(s\right)} \\
\xi n_0 + \frac{\partial P\left(\phi_t, N_t^w\right)}{\partial N_t^w} &= \Sigma_{\hat{s}} \frac{\partial P\left(\phi_t\right)}{\partial \phi_t\left(\hat{s}\right)} \frac{\partial \Gamma\left(\hat{s}\right)}{\partial \phi_{e,t}\left(\hat{s}\right)} \frac{\partial \phi_{e,t}\left(\hat{s}\right)}{\partial m_t} \\
\gamma z k^{\alpha} n^{\gamma-1} &= \xi.
\end{aligned}
$$

By the envelope theorem,

$$
\begin{aligned}
r\frac{\partial P\left(\phi_t\right)}{\partial \phi_t\left(s\right)} &= y\left(s\right) - c\left(\theta_t\left(s\right)\right) + \Sigma_{\hat{s}} \frac{\partial P\left(\phi_t\right)}{\partial \phi_t\left(\hat{s}\right)} \frac{\partial \Gamma\left(\hat{s}\right)}{\partial \phi_t\left(s\right)} + \Sigma_{\hat{s}} \frac{\partial^2 P\left(\phi_t\right)}{\partial \phi_t\left(s\right)\partial \phi_t\left(\hat{s}\right)} \Gamma\left(\hat{s}\right) \\
r\frac{\partial P\left(\phi_t\right)}{\partial N_t^w\left(s\right)} &= \xi + \frac{\partial^2 P\left(\phi_t\right)}{\partial^2 N_t^w\left(s\right)} \dot{N}_t^w - \delta \frac{\partial P\left(\phi_t\right)}{\partial N_t^w\left(s\right)}.
\end{aligned}
$$

We define the marginal value to the planner of an additional agent of type s at time t,

$$
\tilde{V}\left(s;\phi_t\right) = \frac{\partial P\left(\phi_t\right)}{\partial \phi_t\left(s\right)}.
$$

We can formulate the envelope condition above as

$$
r\tilde{V}(s;\phi_t) = y\left(s\right) - c\left(\theta_t\left(s\right)\right) + \Sigma_{\hat{s}} \tilde{V}(\hat{s};\phi_t) \frac{\partial \Gamma\left(\hat{s}\right)}{\partial \phi(s)} + \Sigma_{\hat{s}} \frac{\partial \tilde{V}(s;\phi_t)}{\partial \phi\left(\hat{s}\right)} \Gamma\left(\hat{s}\right).
$$

We also define the marginal value along the optimal trajectory

$$
V_t\left(s\right) = \tilde{V}\left(s;\phi_t\right),
$$

and obtain the time-derivative

$$\frac{\partial V_t(s)}{\partial t} = \Sigma_{\hat{s}} \frac{\partial \tilde{V}(s,\phi_t)}{\partial \phi_t(\hat{s})} \Gamma(\hat{s}).$$

The envelope condition can be further simplified to

$$rV_t\left(s\right) = y\left(s\right) - c\left(\theta_t\left(s\right)\right) + \Sigma_{\hat{s}} V_t\left(\hat{s}\right) \frac{\partial \Gamma\left(\hat{s}\right)}{\partial \phi(s)} + \frac{\partial V_t(s)}{\partial t}.$$

We focus on a stationary planner's problem, which allows us to drop the time subscript and the time derivative from the problem above. Hence,

$$rV\left(s\right) = y\left(s\right) - c\left(\theta\left(s\right)\right) + \Sigma_{\hat{s}} V\left(\hat{s}\right) \frac{\partial \Gamma\left(\hat{s}\right)}{\partial \phi\left(s\right)} \tag{26}$$

$$= y\left(s\right) - ... \tag{27}$$

$$+ \left(V(z,k+1) - V(z,k)\right)(\theta - \delta_k) - C\left(\theta\left(s\right)\right) \tag{28}$$

$$+ \left(V(z+1,k) - V(z,k)\right)\tilde{\mu}^{+}(z) + \left(V(z-1,k) - V(z,k)\right)\tilde{\mu}^{-}(z) \tag{29}$$

$$+ \Sigma_{\hat{s}} V\left(\hat{s}\right) \frac{\partial \Gamma_{\lambda}\left(\hat{s}\right)}{\partial \phi\left(s\right)}. \tag{30}$$

The FOCs with respect to investment and entry become

$$c'\left(\theta_t\left(s\right)\right)\phi_t\left(s\right) = \Sigma_{\hat{s}} \frac{\partial P\left(\phi_t\right)}{\partial \phi_t\left(\hat{s}\right)} \frac{\partial \Gamma\left(\hat{s}\right)}{\partial \theta_t\left(s\right)} = \Sigma_{\hat{s}} V\left(\hat{s}\right) \frac{\partial \Gamma\left(\hat{s}\right)}{\partial \theta\left(s\right)} = \left[V\left(z,k+1\right) - V\left(z,k\right)\right]\phi_t(s) \tag{31}$$

and

$$c_e = \Sigma_{\hat{s}} \frac{\partial P\left(\phi_t\right)}{\partial \phi_t\left(\hat{s}\right)} \frac{\partial \Gamma\left(\hat{s}\right)}{\partial \phi_{e,t}\left(\hat{s}\right)} \frac{\partial \phi_{e,t}\left(\hat{s}\right)}{\partial m_t} = \Sigma_{\hat{s}} V(\hat{s}) g(\hat{s}). \tag{32}$$

Next, we turn to a linear programming problem in which we solve for the optimal set of matches and capital allocations $(\lambda, \lambda_0, k^m)$. We also show that the last term in (30) is equal to the multiplier associated to the constraints of the same linear programming problem.

Optimal Matching

We set up the following linear problem

$$\max_{\lambda\geq 0,\lambda_0\geq 0,k^m} \Sigma_s V(s)\,\Gamma_\lambda(s,\phi;\lambda,\lambda_0,k^m)$$

$$s.t.\quad \Sigma_{\tilde{s}}\lambda(s,\tilde{s})+\lambda_0(s)=1 \quad \forall s$$

$$\Sigma_s\lambda(s,\tilde{s})\,\phi(s)+\lambda_0(\tilde{s})\,\phi(\tilde{s})=\phi(\tilde{s}) \quad \forall\tilde{s}$$

To make progress, we re-arrange the objective function using the definition of Γ_λ,

$$\Sigma_s V(s)\left[\lambda_0(s)\,\phi(s)+\Sigma_{s',s''}\lambda\left(s',s''\right)\mathbb{I}\left\{k^m\left(s',s''\right)=k(s),z\left(s'\right)=z(s)\quad \phi\left(s'\right)\right]\right.$$

$$=\Sigma_s V(s)\left[\lambda_0(s)\,\phi(s)+\Sigma_{s',s''}\left(\frac{\lambda(s',s'')}{2}\mathbb{I}\left\{k^m\left(s',s''\right)=k(s),z\left(s'\right)=z(s)\quad \phi\left(s'\right)\right.\right.\right.$$

$$\left.\left.+\ \frac{\lambda(s'',s')}{2}\mathbb{I}\left\{k^m\left(s',s''\right)=k(s),z\left(s''\right)=z(s)\quad \phi\left(s''\right)\right)\right]$$

$$=\Sigma_s V(s)\left[\lambda_0(s)\,\phi(s)+\Sigma_{s',s''}\left(\frac{\lambda(s',s'')}{2}\phi\left(s'\right)\Sigma_s V(s)\,\mathbb{I}\left\{k^m\left(s',s''\right)=k(s),z\left(s'\right)=z(s)\right.\right.\right.$$

$$\left.\left.+\ \frac{\lambda(s'',s')}{2}\phi\left(s''\right)\Sigma_s V(s)\,\mathbb{I}\left\{k^m\left(s',s''\right)=k(s),z\left(s''\right)=z(s)\quad\right)\right]$$

Imposing feasibility of k^m amounts to restricting the indicators above to be such that either s' is a buyer, or s'' is, or neither. The choice of k^m is equivalent to solving the problem

$$X\left(s',s''\right)=\max\left\{V\left(z',k'+k''\right)+V\left(z'',0\right),V\left(s'\right)+V\left(s''\right),V\left(z',0\right)+V\left(z'',k'+k''\right)\right\}.$$

The objective function thus simplifies to

$$\Sigma_s V(s)\,\lambda_0(s)\,\phi(s)+\Sigma_{s',s''}\frac{\lambda(s',s'')}{2}\phi\left(s'\right)X\left(s',s''\right)$$

Let $\pi(s,\tilde{s})=\frac{\lambda(s,\tilde{s})}{2}\phi(s)$ and $\pi_0(s)=\frac{\lambda_0(s)}{2}\phi(s)$.

We label the value to the matching problem as Q.

$$\begin{aligned} Q(\phi) = \max_{\pi \geq 0, \pi_0 \geq 0} & \Sigma_{s,\tilde{s}} \pi(s,\tilde{s}) X(s,\tilde{s}) + \Sigma_s V(s) \pi_0(s) + \Sigma_{\tilde{s}} V(\tilde{s}) \pi_0(\tilde{s}) \\ & s.t. \Sigma_{\tilde{s}} \pi(s,\tilde{s}) + \pi_0(s) = \frac{\phi(s)}{2} \\ & s.t. \Sigma_s \pi(s,\tilde{s}) + \pi_0(\tilde{s}) = \frac{\phi(\tilde{s})}{2} \end{aligned} \tag{33}$$

Notice that this formulation of the matching problem is analogous to the one in the competitive equilibrium. Let $\mu^a(s)$ and $\mu^b(s)$ be the multipliers attached to the constraints of (33). From the envelope theorem,

$$\frac{\partial Q}{\partial \phi(s)} = \frac{\mu^a(s) + \mu^b(s)}{2}$$

and by the symmetry of $X(\cdot,\cdot)$, $\mu^a(s) = \mu^b(s) = \mu(s)$. Since at the solution,

$$Q(\phi) = \Sigma_s V(s) \Gamma_\lambda(s,\phi;\lambda^*,\lambda_0^*,k^{m,*}),$$

is satisfied at all ϕ, we can differentiate both sides to obtain

$$\Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi(s)} = \mu(s).$$

Characterization

Notice that the Bellman equation, the optimality condition for θ, and the static matching problem are identical to those in the competitive equilibrium. It immediately follows that the competitive equilibrium solves the planner's problem and the equilibrium value and policy functions are the same as the planner's. Last, let ϕ^* be the stationary distribution associated with the planner's problem. The condition $\phi_0 = \phi^*$ guarantees that the economy is stationary.